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CORRELATION OF SHEAR BEHAVIOR OF ICE WITH BIAxIAL STRESS  
RESPONSE

ABSTRACT

At high temperatures and low strain rates ice may be approximately incompressible. We show that if it were truly incompressible, conventional triaxial tests would provide no more information concerning its response to multiaxial loading than would uniaxial tests. However, biaxial tests do provide multiaxial response information. At low temperatures and high strain rates conventional triaxial tests may provide more information than biaxial tests.

INTRODUCTION

Any useful model of the mechanical behavior of ice must describe response to general multiaxial loading. Present knowledge is limited almost entirely to results of uniaxial compression tests, which cannot determine response to general multiaxial loading. This requires tests with three independent components of loading. Since such tests are difficult to perform, we consider how completely multiaxial response is determined by two simpler tests: conventional triaxial and biaxial tests. We find that when response is strongly pressure dependent (at high strain rates and low temperatures), conventional triaxial tests provide more information than biaxial tests. If response is relatively independent of pressure (high temperature and low strain rates), the reverse is true.

While uniaxial tests cannot determine multiaxial response, they can provide a basis for a heuristic construction of multiaxial relations. In a series of papers, Morland and

Spring have constructed multiaxial nonlinear viscoelastic constitutive relations of the differential type, /1/, /2/, and /4/ and of the integral type, /3/ and /5/. An example is the following isotropic incompressible solid relation of the differential type from /4/:

$$\begin{aligned} \dot{\underline{S}} + \underline{S} (\underline{D} + \underline{W}) + (\underline{D} - \underline{W})\underline{S} - \frac{2}{3} \text{tr} (\underline{S} \underline{D}) \underline{1} + \phi \underline{S} \\ = \phi_1 \underline{D} + \phi_2 \underline{D}^2 - \frac{1}{3} (\text{tr} \underline{D}^2) \underline{1} + \omega_1 \left[ \underline{B} - \frac{1}{3} (\text{tr} \underline{B}) \underline{1} \right] \\ + \omega_2 \left[ \underline{B}^2 - \frac{1}{3} (\text{tr} \underline{B}^2) \underline{1} \right] \end{aligned} \quad (1)$$

This relates the tensor quantities, deviatoric stress  $\underline{S}$ ; strain rate  $\underline{D}$ ; rate of rotation  $\underline{W}$ ; and strain  $\underline{B}$ , through the (scalar) response coefficients  $\phi$ ,  $\phi_1$ ,  $\phi_2$ ,  $\omega_1$ ,  $\omega_2$ . These later are material properties which depend in general on  $\underline{S}$ ,  $\underline{B}$ ,  $\underline{D}$ , and their products and/or derivatives. The operator  $\text{tr}$  is the sum of diagonal elements, and  $\underline{1}$  is the unit tensor. It is sufficient to P consider only stress to make our point. Since we will compare compressible with incompressible behavior, it is convenient to use  $\underline{S}$ . This is related to stress  $\underline{\sigma}$ , by ←

$$\underline{S} = \underline{\sigma} + p \underline{1}, \quad (2)$$

where

$$p = -\frac{1}{3} \text{tr} \underline{\sigma} = -\frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3) \quad (3) \quad \leftarrow$$

is the pressure. <sup>sp</sup>Here  $\sigma_1$ ,  $\sigma_2$ , and  $\sigma_3$  are the principal components of stress. The principal components of  $\underline{S}$  are thus

$$\begin{aligned} S_1 &= \frac{2}{3} \sigma_1 - \frac{1}{3} (\sigma_2 + \sigma_3), \quad S_2 = \frac{2}{3} \sigma_2 - \frac{1}{3} (\sigma_1 + \sigma_3), \quad (4) \\ S_3 &= - (S_1 + S_2) \end{aligned}$$

In the following we assume only that the constitutive relations are isotropic. This is equivalent to requiring that response coefficients depend on  $\underline{\sigma}$  only through its invariants  $J_1$ . A complete set of stress invariants is

$$J_1 = p$$

$$J_2 = \frac{1}{2} \text{tr}(\tilde{S}^2) = \frac{1}{3} [\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \quad (5)$$

$$J_3 = \det \tilde{S} = \frac{1}{27} [(2\sigma_1 - \sigma_2 - \sigma_3)(2\sigma_2 - \sigma_3 - \sigma_1)(2\sigma_3 - \sigma_1 - \sigma_2)]$$

For completeness, note that the model leading to equation 1 could be extended to account for compressibility by adding a relation between change in volume and pressure.

#### APPROACH

Our approach is based on the observation that to completely determine response to multiaxial loading we must know each of the response coefficients for all possible sets of values  $(p, J_2, J_3)$ . Let  $p, J_2$  and  $J_3$  be axes in a cartesian coordinate system. The point  $(p, J_2, J_3)$  is in the stress invariants space for a type of test if in that test we may control the stress so that the invariants of the stress tensor have the values  $p, J_2$  and  $J_3$ . Only a true triaxial test can cover the entire space. The more of this space a test can cover, the more completely it can determine multiaxial response.

#### Multiaxial Stress Geometries

First we introduce precise terminology for four possible situations. The case of three independent principal stress  $\sigma_1, \sigma_2, \sigma_3$  will be denoted triaxial stress, abbreviated by TS. Conventional triaxial stress refers to two independent components  $\sigma_1$  and  $\sigma_2$  (say), with  $\sigma_3 = \sigma_2$ , so is correctly described as transversely isotropic stress, abbreviated to TIS. That is, there is an axial stress  $\sigma_1$  with isotropic stress in the normal plane. The case with two independent components  $\sigma_1$  and  $\sigma_2$  and  $\sigma_3 = 0$  is biaxial stress (zero stress in the third direction) abbreviated to BS. Finally, uniaxial stress, abbreviated to US, has one nonzero component  $\sigma_1$ . In summary,

TS (triaxial stress):

3 independent components  $\sigma_1$ ,  $\sigma_2$ , and  $\sigma_3$ ; (6)

TIS (transversely isotropic stress):

2 independent components  $\sigma_1$  and  $\sigma_2$ , with  $\sigma_3 = \sigma_2$ ; (7)

BS (biaxial stress):

2 independent components  $\sigma_1$  and  $\sigma_2$ , with  $\sigma_3 = 0$ ; (8)

US (uniaxial stress):

1 independent component  $\sigma_1$ , with  $\sigma_2 = \sigma_3 = 0$ . (9)

In TS,  $p$ ,  $J_2$ , and  $J_3$  are given by equation 5. In TIS (7) with  $\sigma_3 = \sigma_2$ ,

$$\begin{aligned} S_1 &= \frac{2}{3}(\sigma_1 - \sigma_2), \quad S_3 = S_2 = -\frac{1}{3}(\sigma_1 - \sigma_2), = -\frac{1}{2} S_1, \\ p &= -\frac{1}{3}(\sigma_1 + 2\sigma_2), \quad J_2 = \frac{1}{3}(\sigma_1 - \sigma_2)^2, \quad J_3 = \frac{2}{27}(\sigma_1 - \sigma_2)^3, \end{aligned} \quad (10)$$

Here there is only one independent deviatoric stress  $S_1$ , so even though two independent stresses  $\sigma_1$  and  $\sigma_2$  are applied and varied, only one deviatoric relation is obtained. That is, the deviatoric behavior cannot be determined, noted by Morland /1/.

In BS (8) with  $\sigma_3$ ,

$$\begin{aligned} S_1 &= \frac{2}{3} \sigma_1 - \frac{1}{3} \sigma_2, \quad S_2 = \frac{2}{3} \sigma_2 - \frac{1}{3} \sigma_1, \\ p &= -\frac{1}{3}(\sigma_1 + \sigma_2), \quad J_2 = \frac{1}{3}(\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2), \\ J_3 &= \frac{1}{27}(\sigma_1 + \sigma_2)(\sigma_1 - 2\sigma_2)(2\sigma_1 - \sigma_2), \end{aligned} \quad (11) \quad \leftarrow$$

so  $S_1$  and  $S_2$  are independent, and  $J_2$  and  $J_3$  are independent.

Finally, US (9) with  $\sigma_3 = \sigma_2 = 0$  is a special case of TIS, giving

$$\begin{aligned}
 s_1 &= \frac{2}{3} \sigma_1, \quad s_3 = s_2 = -\frac{1}{3} \sigma_1, = -\frac{1}{2} s_1, \\
 p &= -\frac{1}{3} \sigma_1, \quad J_2 = \frac{1}{3} \sigma_1^2, \quad J_3 = \frac{2}{27} \sigma_1^3.
 \end{aligned}
 \tag{12}$$

As with TIS, only one independent deviatoric relation is obtained.

#### Dependence Domains in Stress Invariants Space

Since in either BS or TIS tests only two components of stress can be varied, it is not possible to determine response independently to all of  $p$ ,  $J_2$ , and  $J_3$ . If response is relatively independent of one of these, we may interpret test results in terms of the other two.

For example, ice is occasionally assumed incompressible, which is equivalent to assuming the response is independent of  $p$ . There is reason to believe this is approximately true at high temperatures and low strain rates (see reference /6/). Response then depends primarily on  $J_2$  and  $J_3$ . Thus we first consider the stress invariant space ( $J_2, J_3$ ). Ice response is pressure dependent in other temperature-strain rate regimes /6/, so we subsequently consider response in the ( $p, J_2$ ) space.

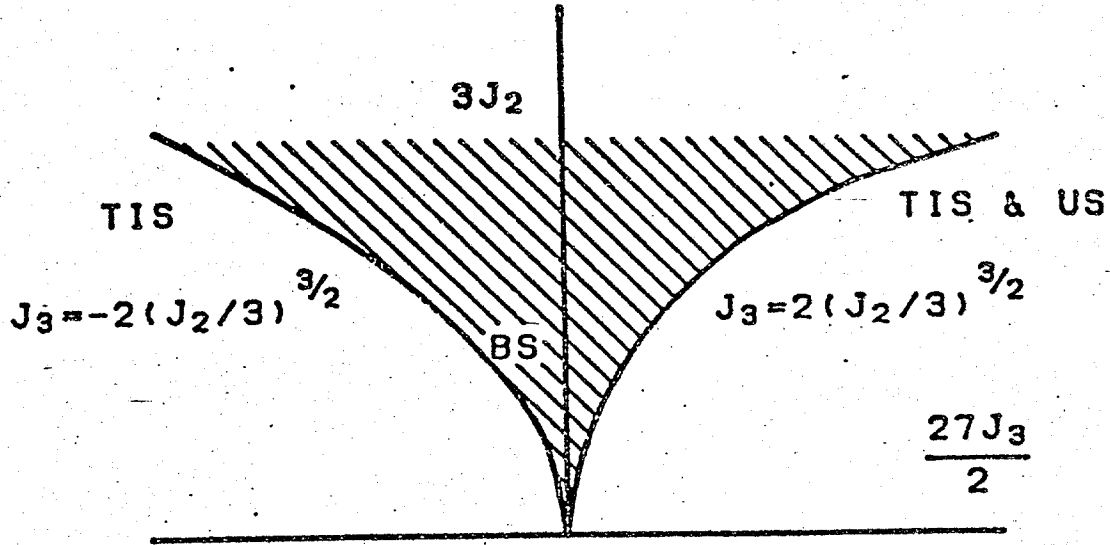
In practice, multiaxial tests will be commonly restricted to compressive stresses  $\sigma_1 \leq 0$ ,  $\sigma_2 \leq 0$ ,  $\sigma_3 \leq 0$ , which limits the domain of the ( $J_2, J_3$ ) plane covered in the US, TIS, and BS geometries. Note that  $J_2 \geq 0$  by definition, but  $J_3$  can be positive or negative. Dependence on  $J_3$  (or  $p$ ) is necessary if compression and tension responses are not symmetric.

#### PRESSURE INDEPENDENCE

First consider the invariants in US (12) with  $\sigma_1 < 0$ . Only a single path

$$J_2 = \frac{1}{3} \sigma_1^2 \geq 0, \quad J_3 = -2 (J_2/3)^{3/2} \leq 0, \tag{13}$$

in the  $(J_2, J_3)$  plane is covered, illustrated in Fig. 1.



Now set

$$\sigma_2 = \eta \sigma_1, \quad \sigma_1 \leq 0, \quad \eta \geq 0, \quad (14)$$

so each positive constant  $\eta$  represents a ray in the compressive  $(\sigma_1, \sigma_2)$  quadrant. US is defined by  $\eta = 0$ . For TIS (10),

$$J_2 = \frac{1}{3} \sigma_1^2 (1-\eta)^2 \geq 0, \quad (15)$$

$$J_3 = \frac{2}{27} \sigma_1^3 (1-\eta)^3 = \pm 2(J_2/3)^{3/2} \text{ as } \eta \gtrless 1$$

Hence the US path (13) is covered for  $\eta < 1$  ( $|\sigma_2| < |\sigma_1|$ ); when  $\eta > 1$ , the alternative branch with  $J_3 > 0$  is covered. This corresponds to uniaxial tension and is also shown in Fig. 1. Thus TIS extends the  $(J_2, J_3)$  domain of US for compressive stress, but both are confined to a curve. Note that the range of  $J_2$  is maximum when  $\eta = 0$ ; that is, for US.

In contrast, BS (11) yields

$$J_2 = \frac{1}{3} \sigma_1^2 (\eta^2 - \eta + 1) \geq 0, J_3 = \frac{1}{27} \sigma_1^3 (2\eta^3 - 3\eta^2 - 3\eta + 2), \quad (16)$$

which we will show are independent for independent  $\sigma_1$  and  $\eta$ .

Eliminate  $\sigma_1$  ( $< 0$ ) between the relation (16) to obtain

$$J_3 = 2k(\eta) \left( \frac{J_2}{3} \right)^{3/2}, k(\eta) = \frac{\left( \eta - \frac{1}{2} \right) \left( \frac{9}{4} - \eta - \frac{1}{2} \right)^2}{\left( \frac{3}{4} + \left( \eta - \frac{1}{2} \right)^2 \right)^{3/2}}. \quad (17)$$

In (15), for any given value of  $J_2$  there are precisely two possible values of  $J_3$ , and these differ only in sign. In contrast, for any given  $J_2$  in (17) there are a range of possible values of  $J_3$ , defined by the range of  $k(\eta)$ .

For  $\sigma_1 > 0$  the sign of  $k$  would be changed and  $\eta \geq 0 \Leftrightarrow \sigma_2 \geq 0$ . At a fixed stress ratio  $\eta > 0$ , hence fixed  $k$ , as  $\sigma_1$  is decreased from zero (17) describes a path in the  $(J_2, J_3)$  plane similar to one of the TIS branches (15), with  $J_3 \geq 0$  as  $k \geq 0$ . Clearly  $k$  is antisymmetric about  $\eta = \frac{1}{2}$  and has the following properties in  $\eta \geq 0$ :

$$k(0) = -1, k\left(\frac{1}{2}\right) = 0, k \rightarrow -1 \text{ as } \eta \rightarrow \infty, \quad (18)$$

$$k_{\min} = k(0) = -1, k_{\max} = k(1) = 1.$$

Thus the minimum value  $k = -1$  occurs when  $\sigma_2 = 0$  (US) and when  $\sigma_1 \rightarrow 0$  at finite  $\sigma_2$  (US in the lateral direction), and the maximum value  $k = 1$  occurs when  $\sigma_2 = \sigma_1$ . Hence a practical range  $0 \leq |\sigma_2| \leq |\sigma_1|$  covers the maximum possible range of  $k$ , and the magnitude of  $\sigma_1$  determines the minimum range of  $J_2$  covered, since  $J_2 \geq \frac{1}{4} \sigma_1^2$  by (16)<sub>1</sub>. The range of  $k$  is duplicated for  $|\sigma_2| \geq |\sigma_1|$  which could provide a test of model consistency. The range limits  $k = \pm 1$  give precisely the two TIS branches (15), corresponding to uniaxial tension and compression, so the BS domain for compressive stress shown in Fig. 1 is the complete sector

between the uniaxial tension and compression branches. In addition, of course, BS provides two deviatoric relations.

#### PRESSURE DEPENDENCE

Suppose that response depends on mean pressure  $p$  and  $J_2$ . Restriction to compressive stress implies  $p \geq 0$ , so results are restricted to the first quadrant of the  $(p, J_2)$  plane. With  $\sigma_2 = \eta \sigma_1 \leq 0$ ,  $\eta \geq 0$ , we have from (12), (10), and (11):

$$\text{US: } p = -\frac{1}{3} \sigma_2 \geq 0, J_2 = \frac{1}{3} \sigma_1^2 = 3p^2, \quad (19)$$

$$\text{TIS: } p = -\frac{1}{3} \sigma_1(1 + 2\eta) \geq 0, J_2 = \frac{1}{3} \sigma_1^2(1 - \eta)^2 = \mu(\eta)p^2, \quad (20)$$

$$\mu(\eta) = 3 \frac{1 - \eta}{1 + 2\eta}^2 \geq 0,$$

$$\text{BS: } p = -\frac{1}{3} \sigma_1(1 + \eta) \geq 0, J_2 = \frac{1}{3} \sigma_1^2(\eta^2 - \eta + 1) = v(\eta)p^2, \quad (21)$$

$$v(\eta) = \frac{3(1 - \eta + \eta^2)}{(1 + \eta)^2} \geq 0.$$

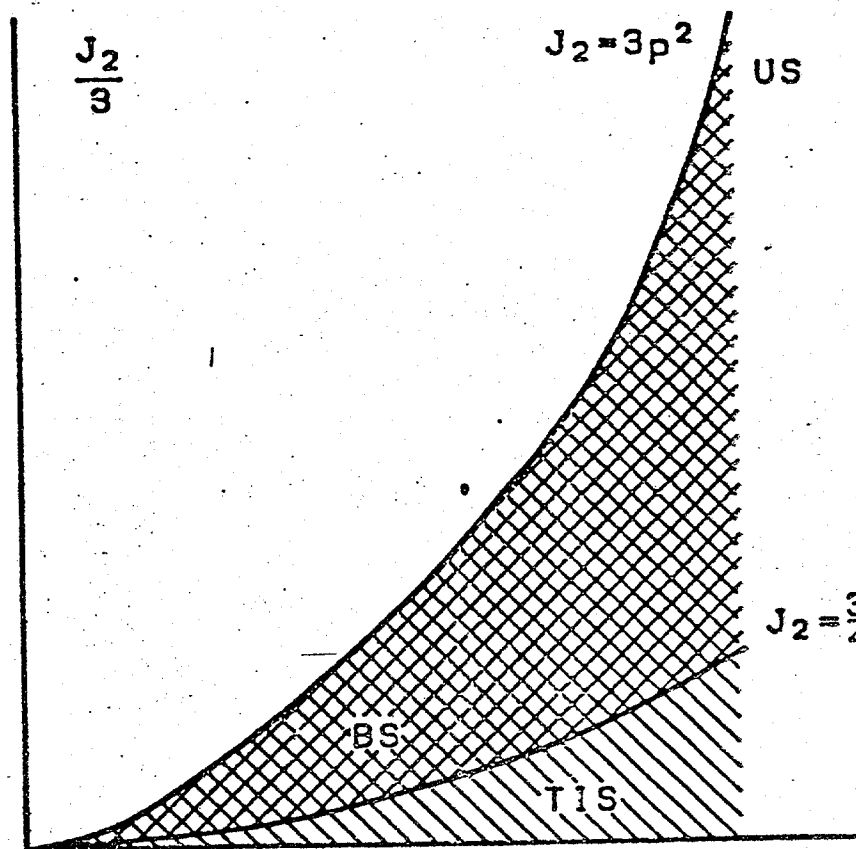
Again US covers only a single curve, shown in Fig. 2, but now both TIS and BS cover two-dimensional domains as  $\mu(\eta)$  and  $v(\eta)$  change continuously with variation of the stress ratio  $\eta$ . While both TIS and BS covered the uniaxial tension branch of the  $(J_2, J_3)$  plane, neither can approach this stress configuration in the  $(p, J_2)$  plane which would require  $p < 0$ .

From (20) and (21),

$$\frac{d\mu}{d\eta} \leq 0 \text{ as } \begin{matrix} 0 \leq \eta < 1 \\ \eta > 1 \end{matrix}, \quad (22)$$

$$\mu(0) = 3, \mu_{\min} = \mu(1) = 0, \mu \rightarrow \frac{3}{4} \text{ as } \eta \rightarrow \infty,$$

and



$$\frac{dv}{d\eta} \leq 0 \text{ as } \begin{matrix} 0 \leq \eta < 1 \\ \eta > 1 \end{matrix}, \quad (23)$$

$$v(0) = 3, \quad v_{\min} = v(1) = \frac{3}{4}, \quad \lambda \rightarrow 3 \text{ as } \eta \rightarrow \infty.$$

Now we see that TIS covers the sector between the US parabola  $J_2 = 3p^2$  and the axis  $J_2 = 0$  for  $0 \leq |\sigma_2| \leq |\sigma_1|$ , shown in Fig. 2, with the range  $0 < \mu < \frac{3}{4}$  duplicated when  $|\sigma_2| > |\sigma_1|$ . However, BS covers only the smaller sector between  $J_2 = 3p^2$  and  $J_2 = \frac{3}{4} p^2$  for  $0 \leq |\sigma_2| \leq |\sigma_1|$  shown in Fig. 2, with the same range  $\frac{3}{4} < \mu < 3$  duplicated when  $|\sigma_2| > |\sigma_1|$ . This is striking reversal of the domains covered by TIS and BS in the  $(J_2, J_3)$  plane, but it is only BS that yields the two deviatoric relations. The excluded domains  $J_2 > 3p^2$  and  $J_2 < \frac{3}{4} p^2$  are of practical significance and cannot be dismissed in model construction. It can be shown that allowing axial tension  $\sigma_1 > 0$  with lateral compression  $\sigma_2 < 0$ ,  $\eta < 0$ ,

extends the domains of both TIS and BS to  $J_2 = 3p^2$  in  $p < 0$ , but the domain  $0 \leq J_2 < \frac{3}{4}$  in  $p > 0$  is still excluded in BS. Both axial tension and lateral tension are required in BS to obtain complete coverage in  $p > 0$ .

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Fig. 1 - Shear invariant domains for compressive stress.

Fig. 2 - Pressure-shear invariant domains  
for compressive stress.